

## **Chaotic Behavior in Shell Models and Shell Maps**

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We study the chaotic behavior of the GOY shell model by measuring the variation of the maximal Lyapunov exponent with the parameter  $\varepsilon$  which determines the nature of the second invariant (the generalized “helicity” invariant). After a Hopf bifurcation, we observe a critical point at  $\varepsilon_c \sim 0.38704$  above which the maximal Lyapunov exponent grows nearly linearly. For high values of  $\varepsilon$  the evolution becomes regular again, which can be explained by a simple analytic argument. A model with few shells shows two transitions. To simplify the model substantially we introduce a shell map which exhibits similar properties as the GOY model.

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**KEY WORDS:** Shell models; turbulence; chaos; Lyapunov exponent; conserved quantities; shell maps.

Shell models of turbulence have been studied intensively by Kadanoff<sup>(1-3)</sup> and many others (for a review see ref. 4). It appears that a lot of properties of highly turbulent flows are nicely captured by those models and numerical computations of shell models are much more tractable than direct simulations of the Navier–Stokes equations. Of particular focus have been studies of intermittency effects where laminar quiescent periods are interrupted by strong intermittency bursts of high energy dissipation. Most shell models follow completely deterministic dynamics but exhibits nevertheless strong intermittent behavior and this is believed to be caused by the internal chaotic dynamics. For the “GOY” shell model,<sup>(5,6)</sup> the strength of this chaos is known to be related to the nature of the second invariant, i.e., the “helicity” invariant.<sup>(1,7)</sup> It is the purpose of this paper to explore the nature of the chaotic behavior as the properties of the second invariant are

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changed. We do this by estimating the variation of the maximal Lyapunov exponent with an external parameter  $\varepsilon$  and find a transition in this exponent where it jumps up from zero to a small but finite value after which it grows linearly. Previous work by Biferale *et al.*<sup>(7)</sup> and Kadanoff *et al.*<sup>(2)</sup> have concentrated on the instability of the “Kolmogorov fixed point” as the nature of the invariant is changed, measured by a parameter  $\varepsilon$  to be introduced below. The first group observed a Hopf bifurcation of the fixed point at  $\varepsilon=0.3843$  turning into a torus at  $\varepsilon=0.3953$  and finally a strange attractor at  $\varepsilon=0.398$ . This was refined by the other group who also studied the detailed eigenvalue spectrum for the unstable modes.<sup>(2)</sup> These studies however only addressed the instability of the stationary state and not the transition to chaotic evolution, which is our purpose here.

Shell models are formed by various truncation techniques of the Navier–Stokes equations.<sup>(4)</sup> The most well-studied model is the “GOY” model of Gledzer–Ohkitani–Yamada.<sup>(1–10)</sup> This model yields corrections to the Kolmogorov theory<sup>(8)</sup> in good agreement with experiments.<sup>(11, 12)</sup> For the “GOY” shell model, wave-number space is divided into  $N$  separated shells each characterized by a wave-number  $k_n = r^n k_0$  ( $r=2$ ), with  $n=1, \dots, N$ . Each shell is assigned a complex amplitude  $u_n$  describing the typical velocity gradient over a scale  $\ell_n = 1/k_n$ . By assuming interactions among nearest and next nearest neighbour shells and phase space volume conservation one arrives at the following evolution equations<sup>(6)</sup>

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = ik_n \left( a_n u_{n+1}^* u_{n+2}^* + \frac{b_n}{2} u_{n-1}^* u_{n+1}^* + \frac{c_n}{4} u_{n-1}^* u_{n-2}^* \right) + f \delta_{n,4} \quad (1)$$

with boundary conditions  $b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0$ .  $f$  is an external, constant forcing, here on the fourth mode.

The coefficients of the non-linear terms must follow the relation  $a_n + b_{n+1} + c_{n+2} = 0$  in order to satisfy the conservation of energy,  $E = \sum_n |u_n|^2$ , when  $f = \nu = 0$ . The constraints still leave a free parameter  $\varepsilon$  so that one can set  $a_n = 1$ ,  $b_{n+1} = -\varepsilon$ ,  $c_{n+2} = -(1-\varepsilon)$ .<sup>(7)</sup> As observed by Kadanoff, one obtains the canonical value  $\varepsilon = 1/2$ , if helicity conservation is also demanded.<sup>(1)</sup> The set (1) of  $N$  coupled ordinary differential equations can be numerically integrated by standard techniques.

To compute the maximal Lyapunov exponent in the GOY model, we introduce the notation  $\mathbf{U} \equiv (\text{Re}(u_1), \text{Im}(u_1), \dots, \text{Re}(u_N), \text{Im}(u_N))$  and  $F_i = dU_i/dt$  and consider the linear variational equations

$$\frac{dz_i}{dt} = \sum_{j=1}^{2N} A_{ij} \cdot z_j \quad i = 1, \dots, 2N \quad (2)$$

for the time evolution of an infinitesimal increment  $\mathbf{z} = \delta\mathbf{U}$ , where

$$A_{nj} \equiv \partial F_n / \partial U_j \tag{3}$$

is the Jacobian matrix of Eqs. (1). The solution for the tangent vector  $\mathbf{z}$  can thus be formally written as  $\mathbf{z}(t_2) = \mathbf{M}(t_1, t_2) \cdot \mathbf{z}(t_1)$ , with  $\mathbf{M} = \exp \int_{t_1}^{t_2} \mathbf{A}(\tau) d\tau$ . A generic tangent vector  $\mathbf{z}(t)$  is projected by the evolution along the eigenvector  $\mathbf{e}^{(1)}$ , belonging to the maximum Lyapunov exponent, i.e.,  $\mathbf{z}(t) = |\mathbf{z}(0)| \mathbf{e}^{(1)} \exp(\lambda_1 t)$  leading to

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{|\mathbf{z}(t)|}{|\mathbf{z}(0)|} \tag{4}$$

where  $\mathbf{z}(0)$  is the initial tangent vector.

Practically, Eqs. (1), (2) are integrated simultaneously over a certain time  $\delta t$ , starting with a normalized tangent vector in a random direction,  $\hat{\mathbf{z}}(0)$ . The increment over time  $\delta t$  in the length of the tangent vector is then  $\delta z_1 = |\mathbf{z}(\delta t)| / |\hat{\mathbf{z}}(0)|$ . Next, the tangent vector is normalized  $\hat{\mathbf{z}}(\delta t) = \mathbf{z}(\delta t) / |\mathbf{z}(\delta t)|$  and this vector is used as a seed for a new integration over the time  $\delta t$ , i.e., propagated forward to  $t = 2\delta t$ . Generalizing this argument we obtain the  $i$ th increment  $\delta z_i = |\mathbf{z}(i\delta t)| / |\hat{\mathbf{z}}((i-1)\delta t)|$  and the maximal Lyapunov exponent is given by (where we now set  $\lambda = \lambda_1$ ):

$$\lambda = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M \frac{\ln(\delta z_i)}{\delta t} \tag{5}$$

We have estimated the maximal Lyapunov exponent this way for the GOY model with standard parameters  $N = 19$ ,  $\nu = 10^{-6}$ ,  $k_0 = 2^{-4}$ ,  $f = (1+i) * 0.005$ . The value of the time increment was varied from  $\delta t = 0.001$  all the way up to  $\delta t = 10$ , in time units set by the choice of parameters and we find that the results are independent of this value, as expected. Figure 1 shows the obtained results, with a variation of  $\varepsilon$  in the interval  $[0, 2]$ . Note the transition to chaos at a critical value  $\varepsilon_c = 0.38704$ .<sup>(13)</sup> Biferale *et al.*<sup>(7)</sup> found a slightly larger value,  $\varepsilon = 0.398$ , for the transition to the attractor at the same parameter values; we do not know the origin of this discrepancy. Magnifying around the transition point we observe a finite but small jump of  $\lambda$  from 0 up to 0.015. This makes the transition first order but we speculate that this jump might disappear in the  $N \rightarrow \infty$  limit. After that the Lyapunov exponent grows more or less linearly with  $\varepsilon$  and reaches a maximum value  $\lambda \sim 0.54$  at  $\varepsilon = 0.92$ . Then it drops sharply down to zero around  $\varepsilon \simeq 1$  after which it raises and drops again. The  $\varepsilon = 1$  is special because at this point the last term in the GOY model, which couples to the two

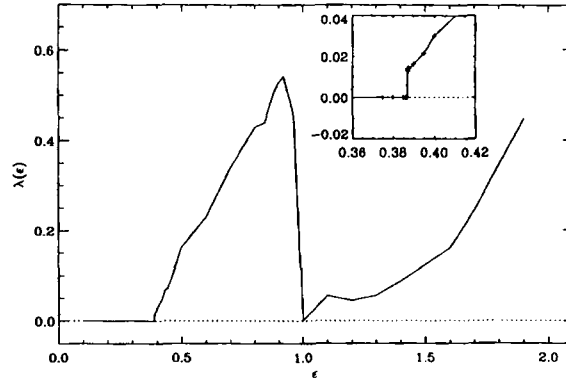


Fig. 1. The variation of the maximal Lyapunov exponent as a function of  $\varepsilon$  for the GOY model equations (1) with the parameters  $N = 19$ ,  $\nu = 10^{-6}$ ,  $k_0 = 2^{-4}$ ,  $f = (1 + i) * 0.005$ . Note the transition to chaos at  $\varepsilon_c = 0.38704$  and the finite, but small jump at this point. The Lyapunov exponent increases nearly linearly but drops back almost to zero at  $\varepsilon = 1$  as explained in the text. The “bump” at  $\varepsilon = 1.1$  is due to the fact that the value of  $\lambda$  has not converged.

previous shells, is zero. This means that for the last three shells, the equations reduce to:

$$\begin{aligned} \left(\frac{d}{dt} + \nu k_{N-2}^2\right) u_{N-2} &= ik_{N-2} \left(u_{N-1}^* u_N^* - \frac{1}{2} u_{N-3}^* u_{N-1}^*\right) \\ \left(\frac{d}{dt} + \nu k_{N-1}^2\right) u_{N-1} &= -ik_{N-2} (u_{N-2}^* u_N^*) \\ \left(\frac{d}{dt} + \nu k_N^2\right) u_N &= 0 \end{aligned} \quad (6)$$

It is seen that  $u_N$  decays exponentially to zero, and therefore we may neglect the right-hand-side of the second equation, which then leads to an exponential decay of  $u_{N-1}$ . Just the same argument can be applied to the behavior of  $u_{N-2}$  up to  $u_4$  where the forcing is applied. We thus expect that the solution essentially converges towards the trivial fixed point  $u_n = 0$ <sup>(14)</sup> and not to the “Kolmogorov fixed point.” This is in accordance with our numerical results, where we see a fast exponential convergence of the modes toward  $u_n = 0$ . Figure 2 shows the variation of the amplitudes  $u_n$  as a function of time. Besides the nice exponential decay, it appears that each mode is successively “triggered” for the decay. This is seen in Fig. 2: the  $n$ th shell is triggered when the absolute value of the amplitude of the  $n + 1$ th shell is a approximately  $5 \cdot 10^4$  and the amplitude of the  $n + 2$ th shell has practically vanished. This means that the equation for the  $n$ th

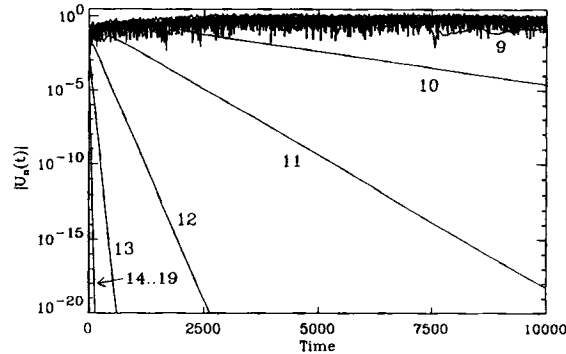


Fig. 2. A logarithmic plot of the long-term dynamics of the absolute value of the shell amplitudes  $|u_n|$  for  $\varepsilon = 1$ . The labeling refers to the shell number and one notes the drastic decay of the higher shell amplitudes.

shell becomes  $(d/dt) u_n = -ik_{n-1}(u_{n-1}^* u_{n+1}^*) - vk_n^2 u_n$ , and just when the  $n + 1$ th amplitude reaches  $\sim 5 \cdot 10^4$  the right-hand-side will be dominated by the viscosity which then forces the  $n$ th amplitude towards zero.

Because the system is forced on the 4th shell, the amplitude of this shell is prevented from going to zero. In fact, when the forcing is included the “trivial fixed point” is  $u_n^* = (0, 0, 0, f/vk_4^2, 0, 0, \dots)$ . Figure 3 shows the variation of the real part of the maximal eigenvalue of the Jacobian as function of  $\varepsilon$  evaluated in this fixed point. We observe that the fixed point is repelling for the standard parameter value  $\varepsilon = 0.5$  but becomes attractive for  $\varepsilon = 1.0$ . This is also the place where the dynamics becomes non-chaotic

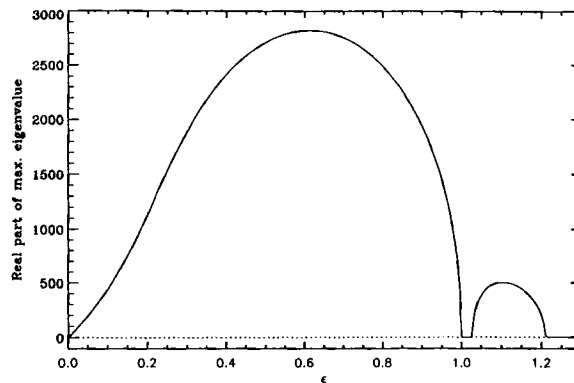


Fig. 3. The value of the real part of the maximal eigenvalue of the Jacobian evaluated in the “trivial fixed point”  $u_n^* = (0, 0, 0, f/vk_4^2, 0, 0, \dots)$ , as a function of  $\varepsilon$ . Note, that the fixed point becomes stable around  $\varepsilon = 1$ .

as indicated by a convergence (although very slow) of the Lyapunov exponent towards zero.

In order to understand the above results further we try to formulate the simplest possible model with the same “symmetries” as the GOY model, a shell map. This map is constructed from the same principles as the GOY model with the difference that the time interval between each update is a full time unit. We then obtain the following map in the real “velocity” variable  $v_n(i)$  of shell  $n$  at integer time  $i$ :

$$v_n(i+1) = v_n(i) + k_n(A_n v_{n+1}(i) v_{n+2}(i) + \frac{B_n}{2} v_{n-1}(i) v_{n+1}(i) + \frac{C_n}{4} v_{n-1}(i) v_{n-2}(i)) - v_1 k_n^2 v_n(i) + f \quad (7)$$

with boundary conditions  $B_1 = B_N = C_1 = C_2 = A_{N-1} = A_N = 0$ . Conservation of the energy  $E = \sum_n v_n(i)^2$  tells us that, for  $v_1 = f_1 = 0$ , the differential  $dE = 2 \sum_n v_n(i) dv_n(i) \simeq 2 \sum_n v_n(i) \cdot (v_n(i+1) - v_n(i)) = 0$  leading to a similar relations between the coefficients as for the GOY model:  $A_n = 1$ ,  $B_n = -\varepsilon_1$ ,  $C_n = -(1 - \varepsilon_1)$ .<sup>(15)</sup> Surprisingly enough, iterations of this model is stable even for very few shells.

To simplify, we start out with a shell map with only  $N = 5$  shells and the following parameters:  $\nu = 10^{-4}$ ,  $k_0 = 2^{-1}$ ,  $f = 0.00005$ . It is immediate to study the eigenvalues of the Jacobian at the fixed point  $u_n(i+1) = u_n(i)$  we locate the fixed point in the stable regime by iterating the map and refining the solution with Newton’s method. This fixed point is a “Kolmogorov fixed point” with a scaling close to  $v_n \sim k_n^{-1/3}$  apart oscillations.<sup>(7)</sup> By varying the parameter  $\varepsilon_1$  in small steps and then using Newton’s method again one can obtain the fixed point where the map becomes unstable. We calculate the eigenvalues of the Jacobian at the fixed point and observe a Hopf-bifurcation (a pair of two complex conjugate eigenvalues escapes from the unit disc in the complex plane) at  $\varepsilon_1 = 0.705081885$ . Here a limit cycle appears. From numerical iteration we observe that the limit cycle has a period of 217.822 (iterations) at  $\varepsilon_1 = 0.70509$ . At this point, the phase of the eigenvalues equals 0.0288458 which is in agreement with the numerically found period since  $2 \cdot \pi / 0.0288458 = 217.820$ . Figure 4a shows the limit cycle for the parameter value  $\varepsilon_1 = 0.73$  (note the somewhat unusual “return plot” where the  $n+1$ th shell amplitude is plotted against the  $n$ th shell amplitude. This gives nicer graphs than a usual return plot). As  $\varepsilon_1$  is increased further a series of standard period doubling bifurcations occur, Fig. 4b shows period four at  $\varepsilon_1 = 0.764$ . Finally, the motion ends up on a strange attractor as shown in Fig. 4c for  $\varepsilon_1 = 0.77$ .

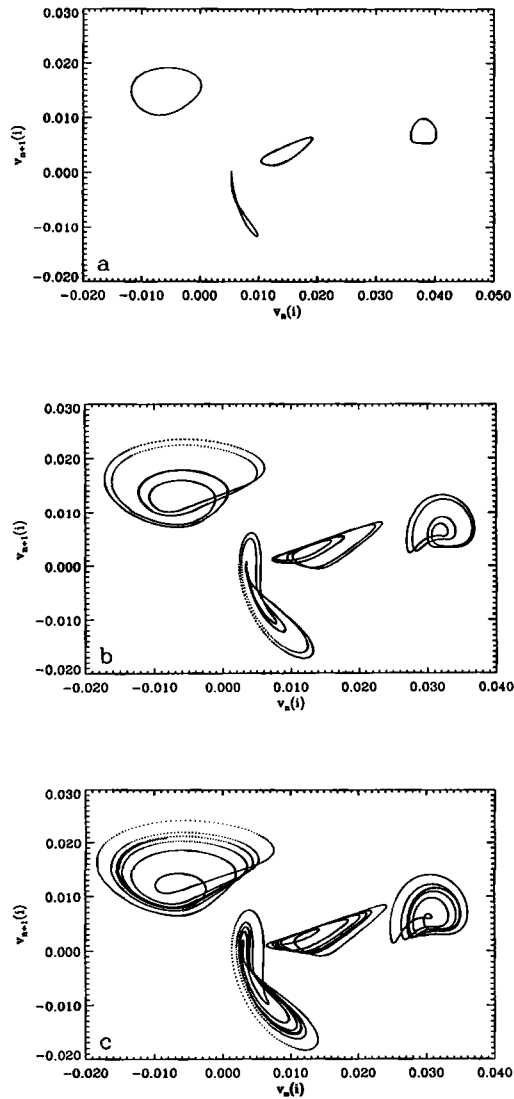


Fig. 4. "Return plots"  $(v_n(i), v_{n+1}(i))$  of the shell map Eq. (7) with the parameters  $N = 5$ ,  $\nu = 10^{-4}$ ,  $k_0 = 2^{-1}$ ,  $f = 0.00005$ . In (a)  $\varepsilon_1 = 0.73$  and we observe limit cycle generated by the Hopf bifurcation; in (b) the second period doubling of the limit cycle is shown at  $\varepsilon_1 = 0.764$  and (c) is the strange attractor for  $\varepsilon_1 = 0.77$ .

To estimate the maximal Lyapunov exponent for the shell map a similar technique as for the GOY model is applied except that now we are dealing with a discrete map. The tangent vector  $\tilde{\mathbf{z}}(i)$  is propagated according to

$$\tilde{z}_k(i+1) = \sum_{j=1}^N \tilde{A}_{kj} \cdot \tilde{z}_j \quad (8)$$

where  $\tilde{A}_{kj}$  is the Jacobian of the map (7)

$$\tilde{A}_{kj} = \partial v_k(i+1) / \partial v_j(i) \quad (9)$$

Again, practically we initiate by a unit tangent vector in a random direction  $\mathbf{z}(0)$  and follow the expansion of the eigenvector. The only difference to the GOY model is that the time increment now is unity so Eq. (5) applies with  $\delta t = 1$ .

Also at low values of  $\varepsilon_1$  there is a transition and Fig. 5 show the variation of the maximal Lyapunov exponent as a function of  $\varepsilon_1$ . For  $\varepsilon_1 \simeq 0$  the model starts out to be chaotic but then the Lyapunov exponent drops and becomes negative around  $\varepsilon_1 \simeq 0.15$ . The exponent approaches zero at  $\varepsilon_1 \simeq 0.705$ . We identify this transition with the appearance of the limit cycle: the distance between two initially close trajectories on the limit cycle will not converge nor diverge. Then the transition to chaos described above occurs at  $\varepsilon_1 = 0.766$ . After that  $\lambda$  grows sharply and drops back to zero at  $\varepsilon_1 \simeq 1$ . It is interesting that there seem to be windows where  $\lambda$  goes to zero; which are like the windows found in the logistic map (see for instance ref. 16).

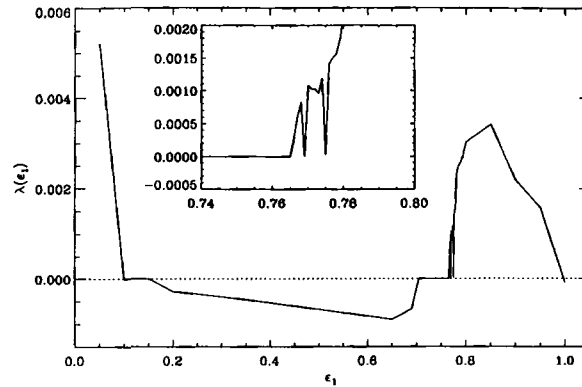


Fig. 5. The maximal Lyapunov exponent versus  $\varepsilon_1$  for the shell map in Eq. (7) for the parameters  $N=5$ ,  $\nu=10^{-4}$ ,  $k_0=2^{-1}$ ,  $f=0.00005$ . Note the two transitions and that the model becomes regular again for  $\varepsilon_1=1$ .



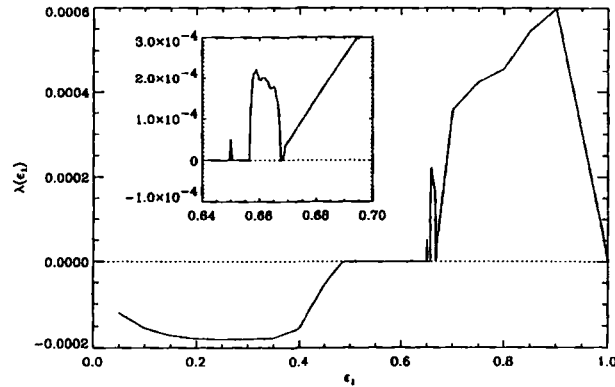


Fig. 6. The maximal Lyapunov exponent versus  $\varepsilon_1$  for the shell map in Eq. (7) for the parameters  $N=11$ ,  $\nu=k_0=2^{-4}$ ,  $f=5 \cdot 10^{-6}$ . Here, there is only one transition at  $\varepsilon_1 \sim 0.66$ .

At  $\varepsilon_1 = 1$  we find again that the system converges towards the fixed point close to zero ( $f/\nu k_1^2, 0, 0, 0, 0$ ) (and not the Kolmogorov fixed point). The eigenvalues of the Jacobian are in this case just  $1 - \nu k_i^2$  ( $i=1, \dots, 5$ ) and thus the fixed point is stable.

It is possible to apply the shell map also for more shells and one can go up to  $N=11$  without divergencies. Figure 6 shows the variation of the maximal Lyapunov exponent as a function of  $\varepsilon_1$ . Here the transition occurs at  $\varepsilon_{1c} = 0.657$ . For this number of shells one can identify an inertial range and below the transition, the Structure functions follow the Kolmogorov predictions, although with the standard oscillations. Above  $\varepsilon_{1c}$ , the motion is intermittent and we again identify windows of stable evolution as for the  $N=5$  case.

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